

# Unbalanced Graph Cuts

Ara Hayrapetyan<sup>1 \*</sup>, David Kempe<sup>2 \*\*</sup>, Martin Pál<sup>3 \*\*\*</sup>, and Zoya Svitkina<sup>1 †</sup>

<sup>1</sup> Dept. of Computer Science, Cornell University. {ara,zoya}@cs.cornell.edu

<sup>2</sup> Dept. of Computer Science, University of Southern California. dkempe@usc.edu

<sup>3</sup> DIMACS Center, Rutgers University. mpal@dimacs.rutgers.edu

**Abstract.** We introduce the MINIMUM-SIZE BOUNDED-CAPACITY CUT (MINSBCC) problem, in which we are given a graph with an identified source and seek to find a cut minimizing the number of nodes on the source side, subject to the constraint that its capacity not exceed a prescribed bound  $B$ . Besides being of interest in the study of graph cuts, this problem arises in many practical settings, such as in epidemiology, disaster control, military containment, as well as finding dense subgraphs and communities in graphs.

In general, the MINSBCC problem is NP-complete. We present an efficient  $(\frac{1}{\lambda}, \frac{1}{1-\lambda})$ -bicriteria approximation algorithm for any  $0 < \lambda < 1$ ; that is, the algorithm finds a cut of capacity at most  $\frac{1}{\lambda}B$ , leaving at most  $\frac{1}{1-\lambda}$  times more vertices on the source side than the optimal solution with capacity  $B$ . In fact, the algorithm’s solution *either* violates the budget constraint, *or* exceeds the optimal number of source-side nodes, but not both. For graphs of bounded treewidth, we show that the problem with unit weight nodes can be solved optimally in polynomial time, and when the nodes have weights, approximated arbitrarily well by a PTAS.

## 1 Introduction

Graph cuts are among the most well-studied objects in theoretical computer science. In the most pristine form of the problem, two given vertices  $s$  and  $t$  have to be separated by removing an edge set of minimum capacity. By a fundamental result of Ford and Fulkerson [16], such an edge set can be found in polynomial time. Since then, many problems have been shown to reduce to graph cut problems, sometimes quite surprisingly (e.g. [19]). One way to view the Min-Cut Problem is to think of “protecting” the sink node  $t$  from the presumably harmful node  $s$  by way of removing edges: the capacity of the cut then corresponds to the cost of edge removals. This interpretation in turn suggests a very natural variant of the graph cut problem: given a node  $s$  and a bound  $B$  on the total edge removal cost, try to “protect” as many nodes from  $s$  as possible, while cutting at most a total edge capacity of  $B$ . In other words, find an  $s$ - $t$  cut of capacity at

---

\* Supported in part by NSF grant CCR-0325453.

\*\* Work done while supported by an NSF graduate fellowship.

\*\*\* Supported by NSF grant EIA 02-05116, and ONR grant N00014-98-1-0589.

† Supported in part by NSF grant CCR-0325453 and ONR grant N00014-98-1-0589.

most  $B$ , minimizing the size of the  $s$ -side of the cut. This is the MINIMUM-SIZE BOUNDED-CAPACITY CUT (MINSBCC) problem that we study.

Naturally, the MINSBCC problem has direct applications in the areas of disaster, military, or crime containment. In all of these cases, a limited amount of resources can be used to monitor or block the edges by which the disaster could spread, or people could escape. At the same time, the area to which the disaster is confined should be as small as possible. For instance, in the firefighter’s problem [6], a fixed small number of firefighters must confine a fire to within a small area, trying to minimize the value of the property and lives inside.

Perhaps even more importantly, the MINSBCC problem arises naturally in the control of epidemic outbreaks. While traditional models of epidemics [4] have ignored the network structure in order to model epidemic diseases via differential equations, recent work by Eubank et al. [7, 9], using highly realistic large-scale simulations, has shown that the graph structure of the social contacts has a significant impact on the spread of the epidemic, and crucially, on the type of actions that most effectively contain the epidemic. If we assume that patient 0, the first infected member of the network, is known, then the problem of choosing which individuals to vaccinate in order to confine the epidemic to a small set of people is exactly the node cut version of the MINSBCC problem.

Besides the obvious connections to the containment of damage or epidemics, the MINSBCC problem can also be used for finding small dense subgraphs and communities in graphs. Discovering communities in graphs has received much attention recently, in the context of analyzing social networks and the World Wide Web [14, 20]. It involves examining the link structure of the underlying graph so as to extract a small set of nodes sharing a common property, usually expressed by high internal connectivity, sometimes in combination with small expansion. We show how to reduce the community finding problem to MINSBCC.

**Our Results.** Formally, we define the MINSBCC problem as follows. Given an (undirected or directed) graph  $G = (V, E)$  with edge capacities  $c_e$ , source and sink nodes  $s$  and  $t$ , as well as a total capacity bound (also called the *budget*)  $B$ , we wish to find an  $s$ - $t$  cut  $(S, \bar{S})$ ,  $s \in S$  of capacity no more than  $B$ , which leaves as few nodes on the source side as possible. We will also consider a generalization in which the nodes are assigned weights  $w_v$ , and the objective is to minimize the total node weight  $\sum_{v \in S} w_v$ , subject to the budget constraint.<sup>1</sup>

We show in Sections 2 and 4.2 that MINSBCC is NP-hard on general graphs with uniform node weights, and on trees with non-uniform node weights. We therefore develop two  $(\frac{1}{\lambda}, \frac{1}{1-\lambda})$ -bicriteria approximation algorithms for MINSBCC, where  $0 < \lambda < 1$ . These algorithms, in polynomial time, find a cut  $(S, \bar{S})$  of capacity at most  $\frac{1}{\lambda}B$ , such that the size of  $S$  is at most  $\frac{1}{1-\lambda}$  times that of  $S^*$ , where  $(S^*, \bar{S}^*)$  is the optimal cut of capacity at most  $B$ . The first algorithm obtains this guarantee by a simple rounding of a linear programming relaxation of MINSBCC. The second one bypasses solving the linear program by running

---

<sup>1</sup> Some of our motivating examples and applications do not specify a sink; this can be resolved by adding an isolated sink to the graph.

a single parametric maximum flow computation and is thus very efficient [17]. It also has a better guarantee: it outputs either a  $(\frac{1}{\lambda}, 1)$ -approximation or a  $(1, \frac{1}{1-\lambda})$ -approximation, thus violating at most one of the constraints by the corresponding factor. The analysis of this algorithm is based on the same linear programming formulation of MINSBCC and its Lagrangian relaxation.

We then investigate the MINSBCC problem for graphs of bounded treewidth in Section 3. We give a polynomial-time algorithm based on dynamic programming to solve MINSBCC optimally for graphs of bounded treewidth with unit node weights. We then extend the algorithm to a PTAS for general node weights.

Section 4 discusses the reductions from node cut and dense subgraph problems to MINSBCC. We conclude with directions for future work in Section 5.

**Related Work.** Minimum cuts have a long history of study and form part of the bread-and-butter of everyday work in algorithms [1]. While minimum cuts can be computed in polynomial time, additional constraints on the size of the cut or on the relationship between its capacity and size (such as its density) usually make the problem NP-hard.

Much recent attention has been given to the computation of *sparse cuts*, partly due to their application in divide-and-conquer algorithms [24]. The seminal work of Leighton and Rao [22] gave the first  $O(\log n)$  approximation algorithm for sparsest and balanced cut problems using region growing techniques. This work was later extended by Garg, Vazirani, and Yannakakis [18]. In a recent breakthrough result, the approximation factor for these problems was improved to  $O(\sqrt{\log n})$  by Arora, Rao, and Vazirani [2].

A problem similar to MINSBCC is studied by Feige et al. [11, 12]: given a number  $k$ , find an  $s$ - $t$  cut  $(S, \overline{S})$  with  $|S| = k$  of minimum capacity. They obtain an  $O(\log^2 n)$  approximation algorithm in the general case [11], and improve the approximation guarantees when  $k$  is small [12].

MINSBCC has a natural maximization version MAXSBCC, where the goal is to maximize the size of the  $s$ -side of the cut instead of minimizing it, while still obeying the capacity constraint. This problem was recently introduced by Svitkina and Tardos [25]. Based on the work of Feige and Krauthgamer [11], Svitkina and Tardos give an  $(O(\log^2 n), 1)$ -bicriteria approximation which is used as a black box to obtain an approximation algorithm for the min-max multiway cut problem, in which one seeks a multicut minimizing the number of edges leaving any one component. The techniques in [25] readily extend to an  $(O(\log^2 n), 1)$ -bicriteria approximation for the MINSBCC problem.

Recently, and independently of our work, Eubank, et al [8] also studied the MINSBCC problem and gave a weaker  $(1 + 2\lambda, 1 + \frac{2}{\lambda})$ -bicriteria approximation.

## 2 Bicriteria Approximation Algorithms

We first establish the NP-completeness of MINSBCC.

**Proposition 1.** *The MINSBCC problem with arbitrary edge capacities and node weights is NP-complete even when restricted to trees.*

**Proof.** We give a reduction from KNAPSACK. Let the KNAPSACK instance consist of items  $1, \dots, n$  with sizes  $s_1, \dots, s_n$  and values  $a_1, \dots, a_n$ , and let the total Knapsack size be  $B$ . We create a source  $s$ , a sink  $t$ , and a node  $v_i$  for each item  $i$ . The node weight of  $v_i$  is  $a_i$ , and it is connected to the source by an edge of capacity  $s_i$ . The sink  $t$  has weight 0, and is connected to  $v_1$  by an edge of capacity 0. The budget for the MINSBCC problem is  $B$ .

The capacity of any  $s$ - $t$  cut is exactly the total size of the items on the  $t$ -side, and minimizing the total node weight on the  $s$ -side is equivalent to maximizing the total value of items corresponding to nodes on the  $t$ -side. ■

Now, we turn our attention to our main approximation results, which are the two  $(\frac{1}{\lambda}, \frac{1}{1-\lambda})$ -bicriteria approximation algorithms for MINSBCC on general graphs. For the remainder of the section, we will use  $\delta(S)$  to denote the capacity of the cut  $(S, \bar{S})$  in  $G$ . We use  $S^*$  to denote the minimum-size set of nodes such that  $\delta(S^*) \leq B$ , i.e.  $(S^*, V \setminus S^*)$  is the optimum cut of capacity at most  $B$ .

The analysis of both of our algorithms is based on the following linear programming (LP) relaxation of the natural integer program for MINSBCC. We use a variable  $x_v$  for every vertex  $v \in V$  to denote which side of the cut it is on, and a variable  $y_e$  for every edge  $e$  to denote whether or not the edge is cut.

$$\begin{aligned}
& \text{Minimize } \sum_{v \in V} x_v \\
& \text{subject to } x_s &= 1 \\
& & x_t &= 0 \\
& & y_e &\geq x_u - x_v \quad \text{for all } e = (u, v) \in E \\
& & \sum_{e \in E} y_e \cdot c_e &\leq B \\
& & x_v, y_e &\geq 0
\end{aligned} \tag{1}$$

## 2.1 Randomized Rounding-Based Algorithm

Our first algorithm is based on randomized rounding of the solution to (1).

---

**Algorithm 1** Randomized LP-rounding algorithm with parameter  $\lambda$

---

- 1: Let  $(x^*, y^*)$  be the optimal solution to LP (1).
  - 2: Choose  $\ell \in [1 - \lambda, 1]$  uniformly at random.
  - 3: Let  $S = \{v \mid x_v^* \geq \ell\}$ , and output  $S$ .
- 

**Theorem 1.** *The Randomized Rounding algorithm (Algorithm 1) outputs a set  $S$  of size at most  $\frac{1}{1-\lambda}$  times the LP objective value. The expected capacity of the cut  $(S, \bar{S})$  is at most  $\frac{1}{\lambda}B$ .*

**Proof.** To prove the first statement of the theorem, observe that for each  $v \in S$ ,  $x_v^* \geq \ell \geq 1 - \lambda$ . Therefore  $\sum_{v \in V} x_v^* \geq \sum_{v \in S} x_v^* \geq (1 - \lambda)|S|$ .

For the second statement, observe that  $\ell$  is selected uniformly at random from an interval of size  $\lambda$ . Furthermore, an edge  $e = (u, v)$  will be cut only if  $\ell$  lies between  $x_u^*$  and  $x_v^*$ . The probability of this happening is thus at most  $\frac{|x_u^* - x_v^*|}{\lambda} \leq \frac{y_e^*}{\lambda}$ . Summing over all edges yields that the expected total capacity

of the cut is at most  $\sum_e \frac{c_e y_e^*}{\lambda} \leq \frac{1}{\lambda} B$ . Notice, that the above algorithm can be derandomized by trying all values  $l = x_v^*$ , since there are at most  $|V|$  of those.

■

## 2.2 A Parametric Flow-Based Algorithm

Next, we show how to avoid solving the LP, and instead compute the cuts directly via a parametric max-flow computation. This analysis will also show that in fact, at most one of the two criteria is approximated, while the other is preserved.

*Algorithm Description:* The algorithm searches for candidate solutions among the parametrized minimum cuts in the graph  $G^\alpha$ , which is obtained from  $G$  by adding an edge of capacity  $\alpha$  from every vertex  $v$  to the sink  $t$  (introducing parallel edges if necessary). Here,  $\alpha$  is a parameter ranging over non-negative values. Observe that the capacity of a cut  $(S, \bar{S})$  in the graph  $G^\alpha$  is  $\alpha|S| + \delta(S)$ , so the minimum  $s$ - $t$  cut in  $G^\alpha$  minimizes  $\alpha|S| + \delta(S)$ .

Initially, as  $\alpha = 0$ , the min-cut of  $G^\alpha$  is the min-cut of  $G$ . As  $\alpha$  increases, the source side of the min-cut of  $G^\alpha$  will contain fewer and fewer nodes, until eventually it contains the single node  $\{s\}$ . All these cuts for the different values of  $\alpha$  can be found efficiently using a single run of the push relabel algorithm. Moreover, the source sides of these cuts form a nested family  $S_0 \supset S_1 \supset \dots \supset S_k$  of sets [17]. ( $S_0$  is the minimum  $s$ - $t$  cut in the original graph, and  $S_k = \{s\}$ ). Our solution will be one of these cuts  $S_j$ .

We first observe that  $\delta(S_i) < \delta(S_j)$  if  $i < j$ ; for if it were not, then  $S_j$  would be a superior cut to  $S_i$  for all values of  $\alpha$ . If  $\delta(S_k) \leq B$ , then, of course,  $\{s\}$  is the optimal solution. On the other hand, if  $\delta(S_0) > B$ , then no solution exists. In all other cases, choose  $i$  such that  $\delta(S_i) \leq B \leq \delta(S_{i+1})$ . If  $\delta(S_{i+1}) \leq \frac{1}{\lambda} B$ , then output  $S_{i+1}$ ; otherwise, output  $S_i$ .

**Theorem 2.** *The above algorithm produces either (1) a cut  $S^-$  such that  $\delta(S^-) \leq B$  and  $|S^-| \leq \frac{1}{1-\lambda}|S^*|$ , or (2) a cut  $S^+$  such that  $\delta(S^+) \leq \frac{1}{\lambda} B$  and  $|S^+| \leq |S^*|$ .*

**Proof.** For the index  $i$  chosen by the algorithm, we let  $S^- = S_i$  and  $S^+ = S_{i+1}$ . Hence,  $\delta(S^-) \leq B \leq \delta(S^+)$ .

First, observe that  $|S^+| \leq |S^*|$ , or else the parametric cut procedure would have returned  $S^*$  instead of  $S^+$ . If  $S^+$  also satisfies  $\delta(S^+) \leq \frac{1}{\lambda} B$ , then we are done. In the case that  $\delta(S^+) > \frac{1}{\lambda} B$ , we will prove that  $|S^-| \leq \frac{1}{1-\lambda}|S^*|$ .

Because  $S^+$  and  $S^-$  are neighbors in our sequence of parametric cuts, there is a value of  $\alpha$ , call it  $\alpha^*$ , for which both are minimum cuts of  $G^{\alpha^*}$ . Applying the Lagrangian Relaxation technique, we remove the constraint  $\sum_e y_e c_e \leq B$  from LP (1) and put it into the objective function using the constant  $\alpha^*$ .

$$\begin{aligned}
& \text{Minimize } \alpha^* \cdot \sum_{v \in V} x_v + \sum_{e \in E} y_e \cdot c_e \\
& \text{subject to } x_s = 1 \\
& \quad x_t = 0 \\
& \quad y_e \geq x_u - x_v \quad \text{for all } e = (u, v) \in E \\
& \quad x_v, y_e \geq 0
\end{aligned} \tag{2}$$

**Lemma 1.** *LP (2) has an integer optimal solution.*

**Proof.** Recall that in  $G^{\alpha^*}$  we added edges of capacity  $\alpha^*$  from every node to the sink. Extend any solution of LP (2) to these edges by setting  $y_e = x_v - x_t = x_v$  for the newly added edge  $e$  connecting  $v$  to  $t$ . We claim that after this extension, the objective function of LP (2) is equivalent to  $\sum_{e \in G^{\alpha^*}} y_e c'_e$ , where  $c'_e$  is the edge capacity in the graph  $G^{\alpha^*}$ . Indeed, this claim follows from observing that the first part of the objective of LP (2) is identical to the contribution that the newly added edges of  $G^{\alpha^*}$  are making towards  $\sum_{e \in G^{\alpha^*}} y_e c'_e$ .

Consider a fractional optimal solution  $(\hat{x}, \hat{y})$  to LP (2) with objective function value  $L^* = \sum_{e \in G^{\alpha^*}} \hat{y}_e c'_e$ . As this is an optimal solution, we can assume without loss of generality that  $y_e = \max(0, x_u - x_v)$  for all edges  $e = (u, v)$ . So if we define  $w_x = \sum_{u, v: x_u \geq x \geq x_v} c'_{uv}$ , then  $L^* = \int_0^1 w_x dx$ .

Also, for any  $x \in (0, 1)$ , we can obtain an integral solution to LP (2) whose objective function value is  $w_x$  by rounding  $\hat{x}_v$  to 0 if it is no more than  $x$ , and to 1 otherwise (and setting  $y_{uv} = \max(0, x_u - x_v)$ ). Since this process yields feasible solutions, we know that  $w_x \geq L^*$  for all  $x$ . On the other hand,  $L^*$  is a weighted average (integral) of  $w_x$ 's, and hence in fact  $w_x = L^*$  for all  $x$ , and any of the rounded solutions is an integral optimal solution to LP (2). ■

Notice that feasible integral solutions to LP (2) correspond to  $s$ - $t$  cuts in  $G^{\alpha^*}$ . Therefore, by Lemma 1, the optimal solutions to LP (2) are the minimum  $s$ - $t$  cuts in  $G^{\alpha^*}$ . In particular,  $S^+$  and  $S^-$  are two such cuts. From  $S^+$  and  $S^-$ , we naturally obtain solutions to LP (2), by setting  $x_v^+ = 1$  for  $v \in S^+$  and  $x_v^+ = 0$  otherwise, with  $y_e^+ = 1$  if  $e$  is cut by  $(S^+, \overline{S^+})$ , and 0 otherwise (similarly for  $S^-$ ). By definition of  $\alpha^*$ , both  $(x^+, y^+)$  and  $(x^-, y^-)$  are then optimal solutions to LP (2). Thus, their linear combination  $(x^*, y^*) = \ell \cdot (x^+, y^+) + (1 - \ell) \cdot (x^-, y^-)$  is also an optimal feasible solution. Choose  $\ell$  such that

$$\ell \cdot \sum_{e \in E} y_e^+ c_e + (1 - \ell) \cdot \sum_{e \in E} y_e^- c_e = B. \quad (3)$$

Such an  $\ell$  exists because our choice of  $S^-$  and  $S^+$  ensured that  $\delta(S^-) \leq B \leq \delta(S^+)$ . For this choice of  $\ell$ , the fractional solution  $(x^*, y^*)$ , in addition to being optimal for the Lagrangian relaxation, also satisfies the constraint  $\sum_e y_e^* c_e \leq B$  of LP (1) with equality, implying that it is optimal for LP (1) as well. Crudely bounding the second term in Equation (3) by 0, we obtain that  $\delta(S^+) = \sum_{e \in E} y_e^+ c_e \leq \frac{B}{\ell}$ .

As we assumed that  $\delta(S^+) > \frac{B}{\lambda}$ , we conclude that  $\ell < \lambda$ . Because  $(x^*, y^*)$  is an optimal solution to LP (1), it provides the lower bound  $\sum_v x_v^* \leq |S^*|$ , and the fact that  $x_v^* \geq (1 - \ell)x_v^-$  now implies that  $|S^-| = \sum_{v \in V} x_v^- \leq \frac{|S^*|}{1 - \ell} \leq \frac{1}{1 - \lambda} \cdot |S^*|$ .

Hence, in this case,  $S^-$  meets the capacity constraint, and exceeds the optimal size by at most a factor of  $\frac{1}{1 - \lambda}$ . ■

Both of the above algorithms can be extended with simple modifications to allow for node weights in addition to edge capacities.

### 3 Bounded Treewidth

As we saw in Section 2, the MINSBCC problem is NP-complete even on trees when both node weights and edge capacities are allowed. However, if all nodes have unit weights, then the problem can be solved in polynomial time for graphs of bounded treewidth, via a dynamic programming algorithm. In order to present the intuition behind our algorithm, we first describe it for trees, and then extend it to graphs of bounded treewidth (see [19] for a review of tree decompositions).

#### 3.1 An Algorithm for Trees

We root the tree at the source node  $s$  and direct all edges away from  $s$ . When all edges have capacity 1, then clearly, only edges incident with  $s$  should be cut. They must include the edge on the unique  $s$ - $t$  path, and in addition, the edges to the roots of the largest subtrees. Choosing these  $B$  edges gives the smallest possible size for the  $s$ -side of the cut.

For the case of general edge capacities, consider the tree  $T_v$  rooted at a node  $v$ , together with the edge  $e_v$  into  $v$ . We define the quantity  $a_v^k$  to be the smallest total capacity of edges in  $T_v$  that must be cut if at most  $k$  nodes of  $T_v$  are to be included in the source side of the cut. Notice that  $a_v^0 = c_{e_v}$ . Also, as the sink must always be excluded, we have  $a_t^k = c_{e_t}$  for all  $k$ .

For a leaf  $v$ , we have  $a_v^0 = c_{e_v}$ , and  $a_v^k = 0$  for  $k > 0$ . For an internal node  $v$  with children  $v_1, \dots, v_d$ , we can either cut the edge  $e_v$  into  $v$ , or otherwise include  $v$  and solve the problem recursively for the children of  $v$ , hence

$$a_v^k = \min(c_{e_v}, \min_{k_1 \geq 0, \dots, k_d \geq 0: \sum k_i = k-1} \sum_i a_{v_i}^{k_i}) \quad \text{for } k > 0, v \neq t.$$

Note that the optimal partition into  $k_i$ 's can be found in polynomial time by a nested dynamic programming subroutine that uses optimal partitions of each  $k$  into  $k_1 \dots k_j$  in order to calculate the optimal partition into  $k_1 \dots k_{j+1}$ .

Once we have computed  $a_s^k$  at the source  $s$  for all values of  $k$ , we simply pick the smallest  $k^*$  such that  $a_s^{k^*} \leq B$ .

#### 3.2 An Algorithm for Graphs with Bounded Treewidth

Recall [19] that a graph  $G = (V, E)$  has treewidth  $\theta$  if there exists a tree  $T$ , and subsets  $V_w \subseteq V$  of nodes associated with each vertex  $w$  of  $T$ , such that:

1. Every node  $v \in V$  is contained in some subset  $V_w$ .
2. For every edge  $e = (u, v) \in E$ , some set  $V_w$  contains both  $u$  and  $v$ .
3. If  $\hat{w}$  lies on the path between  $w$  and  $w'$  in  $T$ , then  $V_w \cap V_{w'} \subseteq V_{\hat{w}}$ .
4.  $|V_w| \leq \theta + 1$  for all vertices  $w$  of the tree  $T$ .

The pair  $(T, \{V_w\})$  is called a *tree decomposition* of  $G$ , and the sets  $V_w$  will be called *pieces*. It can be shown that for any two neighboring vertices  $w$  and  $w'$  of

the tree  $T$ , the deletion of  $V_w \cap V_{w'}$  from  $G$  disconnects  $G$  into two components, just as the deletion of the edge  $(w, w')$  would disconnect  $T$  into two components.

We assume that we are given a tree decomposition  $(T, \{V_w\})$  for  $G$  with treewidth  $\theta$  [5]. To make sure that each edge of the original graph is accounted for exactly once by the algorithm, we partition the set  $E$  by mapping each edge in it to one of the nodes in the decomposition tree. In other words, we associate with each node  $w \in T$  a set  $E_w \subseteq E \cap (V_w \times V_w)$  of edges both of whose endpoints lie in  $V_w$ , such that each edge appears in exactly one set  $E_w$ ; if an edge lies entirely in  $V_w$  for several nodes  $w$ , we assign it arbitrarily to one of them. We will identify some node  $r$  of the tree  $T$  with  $s \in V_r$  as being the *root*, and consider the edges of  $T$  as directed away from  $r$ .

Let  $W \subseteq T$  be the set of nodes in the subtree rooted at some node  $w$ ,  $E_W = \bigcup_{u \in W} E_u$ , and  $V_W = \bigcup_{u \in W} V_u$ . Also, let  $U, U' \subseteq V_w$  be arbitrary disjoint sets of nodes. We define  $a_w^k(U, U')$  to be the minimum capacity of edges from  $E_W$  that must be cut by any set  $S \subseteq V_W$  such that  $S \supseteq U$ ,  $S \cap U' = \emptyset$ , the sink  $t$  is not included in  $S$  (i.e.,  $t \notin S$ ), and  $|S \setminus U| \leq k$ . But for the extension regarding the sets  $U$  and  $U'$ , this is exactly the same quantity we were considering in the case of trees. Also, notice that the minimum size of any cut of capacity at most  $B$  is the smallest  $k$  for which  $a_r^{k-1}(\{s\}, \emptyset) \leq B$ .

Our goal is to derive a recurrence relation for  $a_w^k(U, U')$ . At any stage, we will be taking a minimum over all subsets that meet the constraints imposed by the size  $k$  and the sets  $U, U'$ . We therefore write  $\mathcal{S}_w^k(U, U') = \{S \mid U \subseteq S \subseteq V_w, S \cap U' = \emptyset, t \notin S, |S| \leq k\}$ . The size of  $\mathcal{S}_w^k(U, U')$  is  $O(2^\theta)$ . The cost incurred by cutting edges assigned to  $w$  is denoted by  $\beta_w(S) = \sum_{e \in E_w \cap e(S, V_w \setminus S)} c_e$ , and can be computed efficiently.

If  $w$  is a leaf node, then we can include up to  $k$  additional nodes, so long as the constraints imposed by the sets  $U$  and  $U'$  are not violated. Hence,

$$a_w^k(U, U') = \min_{S \in \mathcal{S}_w^k(U, U')} \beta_w(S).$$

For a non-leaf node  $w$ , let  $w_1, \dots, w_d$  denote its children. We can include an arbitrary subset of nodes, so long as we add no more than  $k$  nodes, and do not violate the constraints imposed by the sets  $U$  and  $U'$ . The remaining additional nodes can then be divided among the children of  $w$  in any way desired. Once we have decided to include (or exclude) a node  $v \in V_w$ , this decision must be respected by all children, i.e., we obtain different sets as constraints for the children. Notice that any node  $v$  contained in the pieces at two descendants of  $w$  must also be in the piece at  $w$  itself by property 3 of a tree decomposition. Also, by the same property, any node  $v$  from  $V_w$  that is not in  $V_{w_i}$  (for some child  $w_i$  of  $w$ ) will not be in the piece at any descendant of  $w_i$ , and hence the information about  $v$  being forbidden or forced to be included is irrelevant in the subtree rooted at  $w_i$ . We hence have the following recurrence:

$$a_w^k(U, U') = \min_{S \in \mathcal{S}_w^k(U, U')} \min_{\{k_i\}: \sum k_i = k - |S \setminus U|} \left( \beta_w(S) + \sum_{i=1}^d a_{w_i}^{k_i}(S \cap V_{w_i}, (V_w \setminus S) \cap V_{w_i}) \right).$$

As before, for any fixed set  $S$ , the minimum over all combinations of  $k_i$  values can be found by a nested dynamic program.

By induction over the tree, we can prove that this recursive definition actually coincides with the initial definition of  $a_w^k(U, U')$ , and hence that the algorithm is correct. The computation of  $a_w^k(U, U')$  takes time  $O(d \cdot k \cdot 2^\theta) = O(n^2 \cdot 2^\theta)$ . For each node, we need to compute  $O(n^2 \cdot 4^\theta)$  values, so the total running time is  $O(n^4 \cdot 8^\theta)$ , and the space requirement is  $O(n^2 \cdot 4^\theta)$ . To summarize, we have proved the following theorem:

**Theorem 3.** *For graphs of treewidth bounded by  $\theta$ , there is an algorithm that finds, in polynomial time  $O(8^\theta n^4)$ , an optimal MINSBCC.*

### 3.3 A PTAS for the node-weighted version

We conclude by showing how to extend the above algorithm to a polynomial-time approximation scheme (PTAS) for MINSBCC with arbitrary node weights.

Suppose we want a  $(1 + 2\epsilon)$  guarantee. Let  $S^*$  denote the optimal solution and  $OPT$  denote its value. We first guess  $W$  such that  $OPT \leq W \leq 2 \cdot OPT$  (test all powers of 2). Next, we remove all *heavy* nodes, i.e. those whose weight is more than  $W$ . We then rescale the remaining node weights  $w_v$  to  $w'_v := \lceil \frac{w_v n}{\epsilon W} \rceil$ . Notice that the largest node weight is now at most  $\frac{n}{\epsilon}$ . Hence, we can run the dynamic programming algorithm on the rescaled graph in polynomial time.

We now bound the cost of the obtained solution, which we call  $S$ . The scaled weight of the solution  $S^*$  is at most  $\sum_{v \in S^*} \lceil \frac{w_v n}{\epsilon W} \rceil \leq \frac{n}{\epsilon W} OPT + n$  (since  $|S^*| \leq n$ ). Since  $S^*$  is a feasible solution for the rescaled problem, the solution  $S$  found by the algorithm has (rescaled) weight no more than that of  $S^*$ . Thus, the original weight of  $S$  is at most  $(OPT + \epsilon W)$ . Considering that  $W \leq 2 \cdot OPT$ , we obtain the desired guarantee, namely that the cost of  $S$  is at most  $(1 + 2\epsilon)OPT$ .

## 4 Applications

### 4.1 Epidemiology and Node Cuts

Some important applications, such as vaccination, are phrased much more naturally in terms of node cuts than edge cuts. Here, each node has a *weight*  $w_v$ , the cost of including it on the  $s$ -side of the cut, and a *capacity*  $c_v$ , the cost of removing (cutting) it from the graph. The goal is to find a set  $R \subseteq V$ , not containing  $s$ , of capacity  $c(R)$  not exceeding a budget  $B$ , such that after removing  $R$ , the connected component  $S$  containing  $s$  has minimum total weight  $w(S)$ .

This problem can be reduced to (node-weighted) MINSBCC in the standard way. First, if the original graph  $G$  is undirected, we bidirect each edge. Now, each vertex  $v$  is split into two vertices  $v_{\text{in}}$  and  $v_{\text{out}}$ ; all edges into  $v$  now enter  $v_{\text{in}}$ , while all edges out of  $v$  now leave  $v_{\text{out}}$ . We add a directed edge from  $v_{\text{in}}$  to  $v_{\text{out}}$  of capacity  $c_v$ . Each originally present edge, i.e., each edge into  $v_{\text{in}}$  or out of  $v_{\text{out}}$ , is given infinite capacity. Finally,  $v_{\text{in}}$  is given node weight 0, and  $v_{\text{out}}$  is given node weight  $w_v$ . Call the resulting graph  $G'$ .

Now, one can verify that (1) no edge cut in  $G'$  ever cuts any originally present edges, (2) the capacity of an edge cut in  $G'$  is equal to the node capacity of a node cut in  $G$ , and (3) the total node weight on the  $s$ -side of an edge cut in  $G'$  is exactly the total node weight in the  $s$  component of the corresponding node cut in  $G$ . Hence an approximation algorithm for MINSBCC carries to node-cuts.

## 4.2 Graph Communities

Identifying “communities” has been an important and much studied problem for social or biological networks, and more recently, the web graph [14, 15]. Different mathematical formalizations for the notion of a community have been proposed, but they usually share the property that a community is a node set with high edge density within the set, and comparatively small expansion.

It is well known [21] that the *densest subgraph*, i.e., the set  $S$  maximizing  $\frac{c(S)}{|S|} := \frac{c(S,S)}{|S|}$  can be found in polynomial time via a reduction to MIN-CUT. On the other hand, if the size of the set  $S$  is prescribed to be at most  $k$ , then the problem is the well-studied *densest  $k$ -subgraph problem* [3, 10, 13], which is known to be NP-complete, with the best known approximation ratio of  $O(n^{1/3-\epsilon})$  [10]. We consider the converse of the densest  $k$ -subgraph problem, in which the density of the subgraph is given, and the size has to be minimized.

The definition of a graph community as the densest subgraph has the disadvantage that it lacks specificity. For example, adding a high-degree node tends to increase the density of a subgraph, but intuitively such a node should not belong to the community. The notion of a community that we consider avoids this difficulty by requiring that a certain fraction of a community’s edges lie inside of it. Formally, let an  $\alpha$ -community be a set of nodes  $S$  with  $\frac{c(S)}{d(S)} \geq \alpha$ , where  $d(S)$  is the sum of degrees of nodes in  $S$ . This definition is a relaxation of one introduced by Flake et al. [14] and is used in [23]. We are interested in finding such communities of smallest size.

The problem of finding the smallest  $\alpha$ -community and the problem of finding the smallest subgraph of a given density have a common generalization, which is obtained by defining a node weight  $w_v$  which is equal to node degree for the former problem and to 1 for the latter. We show how to reduce this general size minimization problem to MINSBCC in an approximation-preserving way. In particular, by applying this reduction to the densest  $k$ -subgraph problem, we show that MINSBCC is NP-hard even for the case of unit node weights.

Given a graph  $G = (V, E)$  with edge capacities  $c_e$ , node weights  $w_v$ , and a specified node  $s \in V$ , we consider the problem of finding the smallest (in terms of the number of nodes) set  $S$  containing  $s$  with  $\frac{c(S)}{w(S)} \geq \alpha$ . (The version where  $s$  is not specified can be reduced to this one by trying all nodes  $s$ .) We modify  $G$  to obtain a graph  $G'$  as follows. Add a sink  $t$ , connect each vertex  $v$  to the source  $s$  with an edge of capacity  $d(v) := \sum_u c_{(v,u)}$ , and to the sink with an edge of capacity  $2\alpha w_v$ . The capacity for all edges  $e \in E$  stays unchanged.

**Theorem 4.** *A set  $S \subseteq V$  with  $s \in S$  has  $\frac{c(S)}{w(S)} \geq \alpha$  if and only if  $(S, \overline{S} \cup \{t\})$  is an  $s$ - $t$  cut of capacity at most  $2c(V) = 2 \sum_{e \in E} c_e$  in  $G'$ .*

Notice that this implies that any approximation guarantees on the size of  $S$  carry over from the MINSBCC problem to the problem of finding communities. Also notice that by making all node weights and edge capacities 1, and setting  $\alpha = \frac{k-1}{2}$ , a set  $S$  of size at most  $k$  satisfies  $\frac{c(S)}{w(S)} \geq \alpha$  if and only if  $S$  is a  $k$ -clique. Hence, the MinSBCC problem is NP-hard even with unit node weights. However, the approximation hardness of CLIQUE does not carry over, as the reduction requires the size  $k$  to be known.

**Proof.** The required condition can be rewritten as  $c(S) - \alpha w(S) \geq 0$ . As

$$2(c(S) - \alpha w(S)) = 2c(V) - (c(S, \bar{S}) + \sum_{v \in \bar{S}} d(v) + 2\alpha w(S)),$$

we find that  $S$  is an  $\alpha$ -community iff  $c(S, \bar{S}) + \sum_{v \in \bar{S}} d(v) + 2\alpha w(S) \leq 2c(V)$ . The quantity on the left is the capacity of the cut  $(S, \bar{S} \cup \{t\})$ , proving the theorem. ■

## 5 Conclusion

In this paper, we present a new graph-theoretic problem called the minimum-size bounded-capacity cut problem, in which we seek to find unbalanced cuts of bounded capacity. Much attention has already been devoted to balanced and sparse cuts [24, 22, 18, 2]; we believe that unbalanced cut problems will pose an interesting new direction of research and will enhance our understanding of graph cuts. In addition, as we have shown in this paper, unbalanced cut problems have applications in disaster and epidemics control as well as in computing small dense subgraphs and communities in graphs. Together with the problems discussed in [11, 12, 25], the MINSBCC problem should be considered part of a more general framework of finding *unbalanced* cuts in graphs.

This paper raises many interesting questions for future research. The main open question is how well the MINSBCC problem can be approximated in a single-criterion sense. At this time, we are not aware of any non-trivial upper or lower bounds for its approximability. The work of [11, 25] implies a  $(\log^2 n, 1)$  approximation — however, it approximates the capacity instead of the size, and thus cannot be used for dense subgraphs or communities. Moreover, obtaining better approximation algorithms will require using techniques different from those in this paper, since our linear program has a large integrality gap.

Further open directions involve more realistic models of the spread of diseases or disasters. The implicit assumption in our node cut approach is that each social contact will always result in an infection. If edges have infection probabilities, for instance based on the frequency or types of interaction, then the model becomes significantly more complex. We leave a more detailed analysis for future work.

**Acknowledgments** We would like to thank Tanya Berger-Wolf, Venkat Guruswami, Jon Kleinberg, and Éva Tardos for useful discussions.

## References

1. R. Ahuja, T. Magnanti, and J. Orlin. *Network Flows*. Prentice Hall, 1993.
2. S. Arora, S. Rao, and U. Vazirani. Expander flows, geometric embeddings and graph partitioning. In *STOC*, 2004.
3. Y. Asahiro, K. Iwama, H. Tamaki, and T. Tokuyama. Greedily finding a dense subgraph. *Journal of Algorithms*, 34, 2000.
4. N. Bailey. *The Mathematical Theory of Infectious Diseases and its Applications*. Hafner Press, 1975.
5. H. L. Bodlaender. A linear time algorithm for finding tree-decompositions of small treewidth. *SIAM J. on Computing*, 25:1305–1317, 1996.
6. M. Develin and S. G. Hartke. Fire containment in grids of dimension three and higher, 2004. Submitted.
7. S. Eubank, H. Guclu, V.S.A. Kumar, M.V. Marathe, A. Srinivasan, Z. Toroczkai, and N. Wang. Modelling disease outbreaks in realistic urban social networks. *Nature*, 429:180–184, 2004.
8. S. Eubank, V. S. A. Kumar, M. V. Marathe, A. Srinivasan, and N. Wang. Structure of social contact networks and their impact on epidemics. *AMS-DIMACS Special Volume on Epidemiology*.
9. S. Eubank, V.S.A. Kumar, M.V. Marathe, A. Srinivasan, and N. Wang. Structural and algorithmic aspects of massive social networks. In *SODA*, 2004.
10. U. Feige, G. Kortsarz, and D. Peleg. The dense  $k$ -subgraph problem. In *STOC*, 1993.
11. U. Feige and R. Krauthgamer. A polylogarithmic approximation of the minimum bisection. *SIAM J. on Computing*, 31:1090–1118, 2002.
12. U. Feige, R. Krauthgamer, and K. Nissim. On cutting a few vertices from a graph. *Discrete Applied Mathematics*, 127:643–649, 2003.
13. U. Feige and M. Seltser. On the densest  $k$ -subgraph problem. Technical report, The Weizmann Institute, Rehovot, 1997.
14. G. Flake, S. Lawrence, C. L. Giles, and F. Coetzee. Self-organization of the web and identification of communities. *IEEE Computer*, 35, 2002.
15. G. Flake, R. Tarjan, and K. Tsioutsoulis. Graph clustering techniques based on minimum cut trees. Technical Report 2002-06, NEC, Princeton, 2002.
16. L. Ford and D. Fulkerson. Maximal flow through a network. *Can. J. Math*, 1956.
17. G. Gallo, M. D. Grigoriadis, and R. E. Tarjan. A fast parametric maximum flow algorithm and applications. *SIAM J. on Computing*, 18:30–55, 1989.
18. N. Garg, V. V. Vazirani, and M. Yannakakis. Approximate max-flow min-(multi)cut theorems and their applications. *SIAM J. on Computing*, 1996.
19. J. Kleinberg and E. Tardos. *Algorithm Design*. Addison-Wesley, 2005.
20. R. Kumar, P. Raghavan, S. Rajagopalan, and A. Tomkins. Trawling the web for emerging cyber-communities. In *WWW*, 1999.
21. E. Lawler. *Combinatorial Optimization: Networks and Matroids*. Holt, Rinehard and Winston, 1976.
22. F.T. Leighton and S. Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *Journal of the ACM*, 46, 1999.
23. F. Radicchi, C. Castellano, F. Cecconi, V. Loreto, and D. Parisi. Defining and identifying communities in networks. *Proc. Natl. Acad. Sci. USA*, 2004.
24. D. Shmoys. Cut problems and their application to divide-and-conquer. In D. Hochbaum, editor, *Approximation Algorithms for NP-hard problems*, pages 192–235. PWD Publishing, 1995.
25. Z. Svitkina and E. Tardos. Min-max multiway cut. In *APPROX*, 2004.