A quick note on the relationship between the VC dimension of a Set Cover set system and the corresponding Hitting Set instance

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Let $(X, \mathcal{C})$ be a Set Cover instance, where $\mathcal{C}$ is a collection of sets $S \subseteq X$. Recall that a Set Cover is a subcollection $\mathcal{D} \subseteq \mathcal{C}$ with $\bigcup_{S \in \mathcal{D}} S = X$. A Hitting Set for $(X, \mathcal{C})$ is a set $H \subseteq X$ such that $H \cap S \neq \emptyset$ for all $S \in \mathcal{C}$.

For a Set Cover instance $(X, \mathcal{C})$, define the corresponding Hitting Set instance by $Y := \{y_S \mid S \in \mathcal{C}\}$, and $T_x := \{y_S \mid S \in \mathcal{C} \text{ and } x \in S\}$, writing $\mathcal{C}_H := \{T_x \mid x \in X\}$. Then it is easy to verify that $\mathcal{D}$ is a set cover for $(X, \mathcal{C})$ if and only if $\{y_S \in Y \mid S \in \mathcal{D}\}$ is a hitting set for $(Y, \mathcal{C}_H)$.

Lemma 1 *If $(X, \mathcal{C})$ has VC-dimension (at most) d, then $(Y, \mathcal{C}_H)$ has VC-dimension at most $2^{d+1} - 1$.*

Proof. We prove the contrapositive. Assume that $(Y, \mathcal{C}_H)$ has VC-dimension at least 2^{d+1}, and let $A \subseteq Y$ be a set of size $|S| = 2^{d+1} =: k$ shattered by $\mathcal{C}_H$. We write $A = \{y_{S_0}, y_{S_1}, \ldots, y_{S_{k-1}}\}$, indexing the elements arbitrarily (but consistently for the rest of the proof).

By the definition of shattering, for each $A' \subseteq A$, there is some $x \in X$ such that $A' = A \cap T_x$. More specifically, this holds for the sets $A'_j = \{y_{S_i} \mid \text{the } j^{\text{th}} \text{ bit of the number } i \text{ is } 1\}$, for $j = 1, \ldots, d+1$; define x_j to be an element such that $A'_j = A \cap T_{x_j}$. By definition of T_{x_j}, the fact that $A'_j = A \cap T_{x_j}$ means that for each $y_S \in A$, we have $y_S \in A'_j$ if and only if $x_j \in S$. And by the definition of A'_j, we get that $x_j \in S_i$ if and only if the j^{th} bit of the number i is 1.

Now define $B = \{x_1, x_2, \ldots, x_{d+1}\}$. We claim that this set is shattered by $\mathcal{C}$. Consider some subset $B' \subseteq B$, and let i be the number corresponding to the bit vector of length $d+1$ for B', i.e., the number whose j^{th} bit is 1 if and only if $x_j \in B'$. We claim that $B' = B \cap S_i$ for this number i, where the sets S_i are still indexed by their ordering in A. To prove this, first consider any $x_j \in B'$. Then, the j^{th} bit of the number i is 1, implying that $y_{S_i} \in A'_j$. But this implies that $x_j \in S_i$, so $x_j \in B \cap S_i$. Next, consider any $x_j \notin B'$. Then, the j^{th} bit of the number i is 0, implying that $y_{S_i} \notin A'_j$. This implies that $x_j \notin S_i$, so $x_j \notin B \cap S_i$. Thus, we have shown that $B' = B \cap S_i$. ∎