CS 671 (Fall 2020) — Assignment 2

Due: 10/02/2020

Read Chapters 1.0, 1.4, 2, 5.6 from the textbook, and optionally Chapters 6.2 and 6.3 from the book by Mitzenmacher/Upfal. Here are the homework problems:

(1) A boxing promoter wants to hold a series of m boxing events. Each event i invites a set S_i of n boxers, who then fight in a sequence of k matches. Each match is scheduled between two distinct boxers from S_i. We assume that each fight is drawn as a uniformly random pair, but the fight does not happen if the same pair has had a fight earlier (i.e., no repeated fights).

The boxers invited into S_i are drawn from a (infinite) population of boxers. Each boxer is either an attacking boxer (with probability 0.5) or a defending boxer (also with probability 0.5). A match is interesting iff it is between an attacking boxer and a defending boxer. An event i is a success iff at least 40% of its matches are interesting. The promoter is interested in the probability that all of the events are successes.

Prove that there exists a constant c such that if $k \geq cm$, then all the events are successes with probability at least $1/2$. (Here, we are assuming that n is chosen large enough so that k distinct matches are in fact possible with n boxers.)

(2) Use the method of conditional expectations to derive a deterministic, polynomial-time, 7/8-approximation algorithm for MAX-3-SAT, and prove its approximation guarantee. Assume that each clause contains exactly 3 literals. Notice that your algorithm should be entirely deterministic, i.e., not make any references to "expectation", "probability", or such (though of course, your proof can use these concepts).

Hint: what you get out may not be the "obvious" algorithm you were expecting at the beginning. If it is, double-check your analysis.

(3) You want to evaluate the integral of functions $f : [0, 1] \to \mathbb{R}$, but you want to do so super-efficiently. So you have come up with the following algorithm: Choose a point $x \in [0, 1]$, and evaluate and output $f(x)$. This is clearly not always correct, so we want to know how far from the correct answer it is in the worst case. If the function f could jump around arbitrarily, you would have no chance of ensuring anything, so we will assume that for all $x, y \in [0, 1]$, you have $|f(x) - f(y)| \leq |x - y|$.

The correct answer is $\int_0^1 f(t)dt$, whereas you output $f(x)$ for one value x. Thus, your absolute error is $|f(x) - \int_0^1 f(t)dt|$; if you choose x from a distribution, it is $\mathbb{E}\left[|f(x) - \int_0^1 f(t)dt|\right]$. Prove the following:

(a) For every deterministic choice of x, there is an input function f such that the absolute error is at least $1/4$.

(b) If you choose $x \in \{1/3, \frac{2}{3}\}$ with probability $1/2$ each, the expected absolute error is at most 1/6.

(c) For every distribution over $x \in [0, 1]$, there is an input function f such that the absolute error is at least 1/8.

If you are really ambitious, you can try to find a distribution and lower bound proof that actually match. (Hint: the value at which they match is $1 - \sqrt{3}/2$.) This is significantly more difficult, and hence not part of the assignment.

(4) Let G be a (directed) graph, s a source, and t a sink. Two players play the following "intrusion" game. Player 1 picks a path P from s to t (possibly randomly), while player 2 picks a set S of r edges of G. Think of player 1 taking P to get from s to t, while player 2 places checkpoints (or patrols) on the edges in S. If $P \cap S = \emptyset$, then player 1 manages to get to t undetected, and wins. If $P \cap S \neq \emptyset$, then player 2 catches player 1, and wins. Player 1 wants to minimize the probability of being caught, while player 2 wants to maximize the probability of catching player 1. Let M be the number of edges in a minimum s-t cut.

(a) Give a (possibly randomized) strategy for player 1 to be caught with probability at most $\min(r/M, 1)$.

(b) Prove that no strategy (randomized or deterministic) can have a probability of being caught strictly less than $\min(r/M, 1)$ against a player 2 who plays perfectly.

Open Problem: Suppose that there are two sinks t_1, t_2. If player 1 manages to get to t_1 undetected, he wins one point. If he gets to t_2 undetected, he wins *two* points. If he is caught, player 2 wins. (It doesn't matter if we call this 0 or 1 points for player 2.) Give an algorithm that computes an optimal randomized strategy for player 1 and/or player 2 (assuming that that player must go first).