CS 670 (Spring 2025) — Assignment 5

Due: 04/28/2025

(1) [10 points]
Consider a standard SET COVER instance with a universe U of size $n = |U|$ and subsets $S_1, \ldots, S_m \subseteq U$. The goal is still to find $J \subseteq \{1, \ldots, m\}$ such that $\bigcup_{j \in J} S_j = U$, and minimize $|J|$. We now add the guarantee that each element occurs in at most k sets, i.e., for all $u \in U$, we have that $|\{j \mid u \in S_j\}| \leq k$. With this extra guarantee, give and analyze (running time, correctness) a polynomial-time k-approximation algorithm for this problem.

(2) [10 points]
Imagine that you have n different sports teams, and each pair of teams played a single game with one winner and one loser. You can then build a directed graph with an edge from u to v if and only if u beat v. So for every pair u, v, you have exactly one of the edges (u, v) or (v, u).

You would now like to rank the teams based on these results, but there are typically cycles, like A beat B, B beat C, C beat D, and D beat A. Your goal is now to remove as few nodes/teams as possible from the graph (and with them all their edges/games) so that the remaining graph has no more cycles (and can therefore be cleanly ranked).

Give and analyze a polynomial-time 3-approximation algorithm for this problem.

Hint: First, you might want to prove the following lemma: if a graph G like the one we described above contains no cycles of length 3, then it contains no cycles at all. Then, Problem 1 on this assignment could be really helpful.

(3) [10 points]
You are given a universe U of n elements. For each $i = 1, \ldots, m$, you are given two subsets $S_i, T_i \subseteq U$. Your goal is to select exactly one of S_i, T_i for each i, while minimizing the size of the union of the sets selected. As an application, imagine that you need to do m jobs, and job i can be performed either by team S_i or by team T_i. Your goal is to hire as few people total as possible, while ensuring that all jobs can be performed with subsets of the people you hired.

Give (and analyze) a polynomial time 2-approximation algorithm for this problem. (Hint: formulate an LP and round it appropriately. Though other techniques may also work.)

(4) [10 points]
We revisit the VERTEX COVER problem with vertex weights we saw in class. Now suppose that in addition to the input graph with vertex weights, someone helpfully computed for you a graph coloring with k colors: that is, an assignment of colors to the vertices such that no two adjacent vertices have the same color (and using at most k distinct colors total). This graph coloring may not be optimal, but that doesn't matter.

Give (and analyze) a polynomial time $(2 - 2/k)$-approximation algorithm for this problem. (So the fewer colors your helper used, the better your approximation guarantee will be.)

Hint: it is strongly recommended that you use the following lemma, which you do not have to prove yourself.

Lemma 1 *The* VERTEX COVER *LP always has a half-integral optimal solution, i.e., a solution in which each variable x_v has value in $\{0, \frac{1}{2}, 1\}$. Furthermore, such a (half-integral, optimal) solution can be found in polynomial time.*

(5) [0 points]
Chocolate Problem (1 chocolate bar):

You are given an undirected graph G with edge costs $c_e \geq 0$, a source s and sink t, and a budget B for cutting edges. Your goal is to produce an s-t cut $(S, \overline{S})$ of capacity at most B such that the size of the s-side is as small as possible, i.e., minimizing $|S|$ subject to $\sum_{e=(u,v):u \in S, v \notin S} c_e \leq B$.

Give and analyze a polynomial-time $(2, 2)$-bicriteria approximation algorithm for this problem. This means that your algorithm outputs an s-t cut of capacity at most $2B$, and its s-side is at most twice as large as the smallest possible s-side of any s-t cut of capacity at most B. (In other words, the algorithm, even though it gets to cut more edges, is compared only against solutions that cut fewer/lighter edges.)

(**Note:** This is by far the easiest chocolate problem all semester. It may even be easier than some of the regular problems on this assignment.)